A CONSENSUS MODEL FOR GROUP DECISION MAKING IN UNBALANCED FUZZY LINGUISTIC CONTEXTS

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Abstract

The aim of this paper is to present a consensus model for group decision making problems with unbalanced linguistic information, i.e., assuming that the preferences are assessed on linguistic term sets whose terms are not symmetrically and uniformly distributed. This consensus model is based on both a fuzzy linguistic methodology to deal with unbalanced linguistic term sets and consensus criteria. Additionally, it presents a feedback mechanism to help experts for reaching a high consensus in decision making processes.

Keywords: fuzzy linguistic modelling, group decision making, consensus.

1 INTRODUCTION

In Group Decision Making (GDM) problems there are a set of alternatives in order to solve a problem and a group of experts trying to achieve a common solution. To solve these problems, the experts are faced by applying two processes before obtaining a final solution [4, 8]: *the consensus process* and *the selection process*. The consensus process is defined as a dynamic and iterative group discussion process, coordinated by a moderator helping experts to bring their opinions closer. If the consensus level is lower than a specified threshold the moderator would urge experts to discuss their opinions further in an effort to bring them closer. Otherwise, the moderator would apply the selection process which consists in obtaining the final solution to the problem from the opinions expressed by the experts. In this framework, an important question is how to substitute the actions of the moderator in order to automatically model the whole consensus process [4].

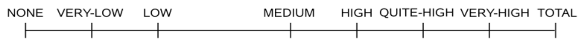

Figure 1: Example of an unbalanced linguistic term set of 8 labels

On the other hand, many GDM problems based on fuzzy linguistic approaches use symmetrically and uniformly distributed linguistic term sets [1, 3, 4, 10], i.e., assuming the same discrimination levels on both sides of mid linguistic term. However, there exist problems that need to assess their variables with linguistic term sets that are not uniformly and symmetrically distributed [2, 7, 9]. This type of linguistic term sets are called [2, 7] *unbalanced fuzzy linguistic term sets* and its use requires to define new methods of management of linguistic information (see Figure 1).

The aim of this paper is to present a consensus model for GDM problems defined in an unbalanced fuzzy linguistic context. In [2, 7] a methodology to manage unbalanced fuzzy linguistic information was presented which used hierarchical linguistic contexts [6] based on the linguistic 2-tuple computational model [5]. However, this methodology can only represent unbalanced linguistic term sets when there exists a level with an adequate granularity to represent the subset of linguistic terms on the left of the mid linguistic term and a level with an adequate granularity to represent the subset of linguistic terms on the right of the mid linguistic term. Thus, we present a new fuzzy linguistic methodology that can represent unbalanced fuzzy linguistic information when the above conditions are not satisfied. As part of the consensus model, a feedback mechanism substituting the figure of the moderator is given to help experts change their opinions on the alternatives in order to obtain the highest degree of consensus possible. It consists of simple and easy rules generating recommendations in the discussion process. Moreover, this model is based on two types of consensus criteria, *consensus degrees* evaluating the agreement of all the experts, and *proximity measures* evaluating the distance between experts' individual opinions and the group or collective opinion which is also used in the feedback mechanism to guide the direction of the changes in experts' opinions in order to increase the consensus degrees.

To do so, the paper is set out as follows. In Section 2, we present the new fuzzy linguistic methodology to manage unbalanced fuzzy linguistic information. Section 3 introduces the new consensus model for GDM problems in an unbalanced fuzzy linguistic context. In Section 4, a practical example is given to illustrate the application of the consensus model. Finally, some concluding remarks are pointed out in Section 5.

2 A NEW FUZZY LINGUISTIC METHODOLOGY TO MANAGE UNBALANCED FUZZY LINGUISTIC INFORMATION

In this section, we make a review of the 2-tuple fuzzy linguistic representation model [5] and the concept of hierarchical linguistic contexts [6] in order to present the new methodology to manage unbalanced fuzzy linguistic information.

2.1 THE 2-TUPLE FUZZY LINGUISTIC REPRESENTATION MODEL

The 2-tuple fuzzy linguistic representation model [5] is based on the concept of *symbolic translation* and represents the linguistic information by means of a pair of values, (s, α), where s is a linguistic label and α is a numerical value that represents the value of the symbolic translation.

Definition 1. *[5] Let β be the result of an aggregation of the indexes of a set of labels assessed in a linguistic term set $S = \{s_0, s_1, \ldots, s_{g-1}, s_g\}$, i.e., the result of a symbolic aggregation operation. $\beta \in [0, g]$, being $g + 1$ the cardinality of S. Let $i = round(\beta)$ and $\alpha = \beta - i$ be two values, such that, $i \in [0, g]$ and $\alpha \in [-0.5, 0.5)$, then α is called a symbolic translation.*

This model defines a set of transformation functions to manage the linguistic information expressed by linguistic 2–tuples.

Definition 2. *Let S be a linguistic term set and $\beta \in [0, g]$ a value supporting the result of a symbolic aggregation operation, then the 2–tuple that expresses the equivalent information to β is obtained with the following function $\Delta : [0, g] \longrightarrow S \times [-0.5, 0.5)$:*

$$\begin{array}{l} \Delta(\beta) = (s_i, \alpha) \\ i = round(\beta) \\ \alpha = \beta - i \end{array} \tag{1}$$

where "round" is the usual round operation, s_i has the closest index label to "β" and "α" is the value of the symbolic translation.

Finally, for all Δ there exists Δ^{-1}, defined as $\Delta^{-1}(s_i, \alpha) = i + \alpha = \beta$.

2.2 HIERARCHICAL LINGUISTIC CONTEXTS

A *Linguistic Hierarchy* is a set of levels, where each level represents a linguistic term set with different granularity from the remaining levels of the hierarchy [6]. Each level is denoted as $l(t, n(t))$, where t is a number indicating the level of the hierarchy, and $n(t)$ is the granularity of the linguistic term set of t. A graphical example of a linguistic hierarchy is shown in Figure 2.

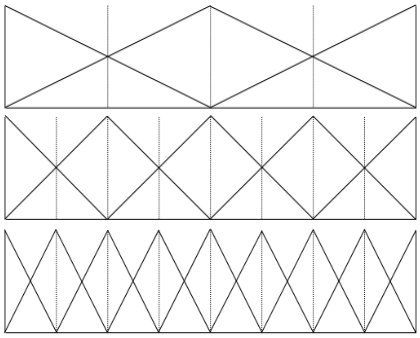

Figure 2: Linguistic hierarchy of 3, 5 and 9 labels

The levels belonging to a linguistic hierarchy are ordered according to their granularity, i.e., for two consecutive levels t and $t+1$, $n(t+1) > n(t)$. Then, a linguistic hierarchy LH can be defined as the union of all levels t: $LH = \bigcup_t l(t, n(t))$.

Given a LH, we denote as $S^{n(t)}$ the linguistic term set of LH corresponding to the level t of LH characterized by a granularity of uncertainty $n(t)$: $S^{n(t)} = \{s_0^{n(t)}, \ldots, s_{n(t)-1}^{n(t)}\}$. Furthermore, the linguistic term set of the level $t+1$ is obtained from its predecessor as: $l(t, n(t)) \rightarrow l(t+1, 2 \cdot n(t) - 1)$.

Transformation functions between labels from different levels to accomplish processes of computing with words in multigranular linguistic information contexts without loss of information were defined in [6].

Definition 3. [6] *Let* $LH = \bigcup_t l(t, n(t))$ *be a linguistic hierarchy whose linguistic term sets are denoted as* $S^{n(t)} = \{s_0^{n(t)}, \ldots, s_{n(t)-1}^{n(t)}\}$*, and let us consider the 2-tuple fuzzy linguistic representation. The transformation function from a linguistic label in level t to a label in level* t' *is defined as* $TF_{t'}^t : l(t, n(t)) \longrightarrow l(t', n(t'))$ *such that*

$$TF_{t'}^t(s_i^{n(t)}, \alpha^{n(t)}) = \Delta_{t'}\left(\frac{\Delta_t^{-1}(s_i^{n(t)}, \alpha^{n(t)}) \cdot (n(t') - 1)}{n(t) - 1}\right). \quad (2)$$

2.2.1 An Unbalanced Fuzzy Linguistic Representation Model

The procedure to represent unbalanced fuzzy linguistic information defined in [2, 7] works as follows:

1. Find a level t^- of LH to represent the subset of linguistic terms S_{un}^L on the left of the mid linguistic term of S_{un}.
2. Find a level t^+ of LH to represent the subset of linguistic terms S_{un}^R on the right of the mid linguistic term of S_{un}.
3. Represent the mid term of S_{un} using the mid terms of the levels t^- and t^+.

The problem appears when there does not exist a level t^- or t^+ in LH to represent S_{un}^L or S_{un}^R, respectively. Then, we propose to overcome this problem by applying the following algorithm, which is defined assuming that there does not exist t^-, as it happens with the unbalanced fuzzy linguistic term set given in Figure 1:

1. Represent S_{un}^L:
   (a) Identify the mid term of S_{un}^L, called S_{mid}^L.
   (b) Find a level t_2^- of the left sets of LH^L to represent the left term subset of S_{un}^L, where LH^L represents the left part of LH.
   (c) Find a level t_2^+ of the right sets of LH^L to represent the right term subset of S_{un}^L.
   (d) Represent the mid term S_{mid}^L using the levels t_2^- and t_2^+.
2. Find a level t^+ of LH to represent the subset of linguistic terms S_{un}^R
3. Represent the mid term of S_{un} using the levels t^+ and t_2^+.

For example, applying this algorithm the representation of the unbalanced fuzzy linguistic term set $S_{un} = \{N, VL, L, M, H, QH, VH, T\}$ shown in Figure 1 with the linguistic hierarchy LH shown in Figure 2 would be as it is shown in Figure 3. In this example,

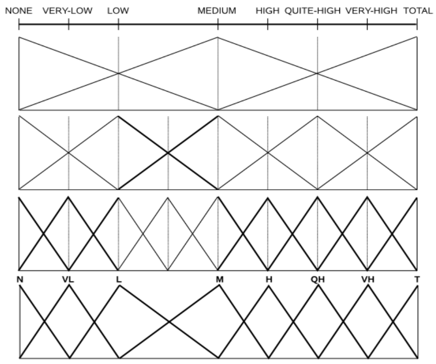

Figure 3: Representation for an unbalanced term set of 8 labels

- $S_{un}^L = \{N, VL, L\}$,
- $S_{mid}^L = L$,
- $LH^L = \{s_0^{n(1)}\} \bigcup \{s_0^{n(2)}, s_1^{n(2)}\} \bigcup \{s_0^{n(3)}, s_1^{n(3)}, s_2^{n(3)}, s_3^{n(3)}\}$.

Thus, we have that $t_2^- = 3$, $t_2^+ = 2$, $S_{mid}^L = L$ is represented using both levels, 3 and 2, and the mid term of S_{un} is represented using the levels 2 and 3.

2.2.2 An Unbalanced Fuzzy Linguistic Computational Model

In any fuzzy linguistic approach we need to define a computational model to manage and aggregate linguistic information. As in [3, 5], we have to define three types of computation operators to deal with unbalanced fuzzy linguistic information: comparison operator, negation operator and aggregation operator. In a unbalanced linguistic context, previously to carry out any computation task of unbalanced fuzzy linguistic information we have to choose a level $t' \in \{t^-, t_2^-, t^+, t_2^+\}$, such that $n(t') = max\{n(t^-), n(t_2^-), n(t^+), n(t_2^+)\}$:

1. *An unbalanced linguistic comparison operator:* The comparison of linguistic information represented by two unbalanced linguistic 2-tuples $(s_k^{n(t)}, \alpha_1)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, and $(s_l^{n(t)}, \alpha_2)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$ is similar to the usual comparison of two 2-tuples but acting on the values $TF_{t'}^t(s_k^{n(t)}, \alpha_1) = (s_v^{n(t')}, \beta_1)$ and $TF_{t'}^t(s_l^{n(t)}, \alpha_2) = (s_w^{n(t')}, \beta_2)$. Then, we have:
   - if $v < w$ then $(s_v^{n(t')}, \beta_1)$ is smaller than $(s_w^{n(t')}, \beta_2)$.
   - if $v = w$ then

(a) if $\beta_1 = \beta_2$ then $(s_v^{n(t')}, \beta_1)$, $(s_w^{n(t')}, \beta_2)$ represent the same information.
(b) if $\beta_1 < \beta_2$ then $(s_v^{n(t')}, \beta_1)$ is smaller than $(s_w^{n(t')}, \beta_2)$.
(c) if $\beta_1 > \beta_2$ then $(s_v^{n(t')}, \beta_1)$ is bigger than $(s_w^{n(t')}, \beta_2)$.

2. *An unbalanced linguistic 2-tuple negation operator.* Let $(s_k^{n(t)}, \alpha)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$ be an unbalanced linguistic 2-tuple, then:

$$NEG(s_k^{n(t)}, \alpha) = Neg(TF_{t''}^{t}(s_k^{n(t)}, \alpha)), \qquad (3)$$

where $t'' \in \{t^-, t_2^-, t^+, t_2^+\}$ and $Neg(s_i, \alpha) = \Delta(g - \Delta^{-1}(s_i, \alpha))$.

3. *An unbalanced linguistic aggregation operator.* To aggregate unbalanced fuzzy linguistic information by means of its representation in a LH, we use the $LOWA_{un}$ operator, which is an extension of the Linguistic Ordered Weighted Averaging operator [3], and is defined as follows:

Definition 4. *Let $\{(a_1, \alpha_1), \ldots, (a_m, \alpha_m)\}$ be a set of unbalanced assessments to aggregate, then the $LOWA_{un}$ operator ϕ_{un} is defined as:*

$$\phi_{un}\{(a_1, \alpha_1), \ldots, (a_m, \alpha_m)\} = W \cdot B^T =$$
$$C_{un}^m\{w_k, b_k,\ k = 1, \ldots, m\} =$$
$$w_1 \otimes b_1 \oplus (1 - w_1) \otimes C_{un}^{m-1}\{\beta_h, b_h,\ h = 2, \ldots, m\}$$

where $b_i = (a_i, \alpha_i) \in (S_{un} \times [-0.5, 0.5))$, $W = [w_1, \ldots, w_m]$, is a weighting vector, such that, $w_i \in [0, 1]$ and $\sum_i w_i = 1$, $\beta_h = \frac{w_h}{\sum_2^m w_k}$, $h = 2, \ldots, m$, and B is the associated ordered unbalanced 2-tuple vector. Each element $b_i \in B$ is the i-th largest unbalanced 2-tuple in the collection $\{(a_1, \alpha_1), \ldots, (a_m, \alpha_m)\}$, and C_{un}^m is the convex combination operator of m unbalanced 2-tuples. If $w_j = 1$ and $w_i = 0$ with $i \neq j\ \forall i, j$ the convex combination is defined as: $C_{un}^m\{w_i, b_i,\ i = 1, \ldots, m\} = b_j$. And if $m = 2$ then it is defined as:

$$C_{un}^2\{w_l, b_l,\ l = 1, 2\} = w_1 \otimes b_j \oplus (1 - w_1) \otimes b_i =$$
$$= TF_t^{t'}(s_k^{n(t')}, \alpha)$$

where $(s_k^{n(t')}, \alpha) = \Delta(\lambda)$ and $\lambda = \Delta^{-1}(TF_{t'}^{t}(b_i)) + w_1 \cdot (\Delta^{-1}(TF_{t'}^{t}(b_j)) - \Delta^{-1}(TF_{t'}^{t}(b_i)))$, $b_j, b_i \in (S_{un} \times [-0.5, 0.5))$, $(b_j \geq b_i)$, $\lambda \in [0, n(t') - 1]$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$.

In [11] it was defined an expression to obtain W by means of a fuzzy linguistic non-decreasing quantifier Q [12]:

$$w_i = Q(i/m) - Q((i-1)/m), \quad i = 1, \ldots, m. \quad (4)$$

3 A CONSENSUS MODEL FOR GDM PROBLEMS DEFINED IN UNBALANCED FUZZY LINGUISTIC CONTEXTS

In this section we present a consensus model defined for GDM problems defined in unbalanced fuzzy linguistic contexts. A GDM problem based on preference relations is classically defined as a decision situation where there are a set of experts, $E = \{e_1, \ldots, e_m\}$ $(m \geq 2)$, and a finite set of alternatives, $X = \{x_1, \ldots, x_n\}$ $(n \geq 2)$, and each expert e_i provides his/her preferences about X by means of a preference relation, $P_{e_i} \subset X \times X$, where the value $\mu_{P_{e_i}}(x_l, x_k) = p_i^{lk}$ is interpreted as the preference degree of the alternative x_l over x_k for e_i. In this paper, we deal with GDM problems defined in unbalanced fuzzy linguistic contexts, i.e., GDM problems where the experts e_i express their preferences relations $P_{e_i} = (p_i^{lk})$ on the set of alternatives X using a linguistic term set that is not uniformly and symmetrically distributed, S_{un}.

This consensus model presents the following main characteristics:

1. It is designed to guide the consensus process of unbalanced fuzzy linguistic GDM problems.

2. It uses a new methodology to manage unbalanced fuzzy linguistic information.

3. It is based on two consensus criteria: *consensus degrees* and *proximity measures*. The first ones are used to measure the agreement amongst all the experts, while the second ones are used to learn how close the collective and individual expert's preference are. Both consensus criteria are calculated at three different levels: pair of alternatives, alternatives and relation. It will allow us to know the current state of consensus from different viewpoints, and therefore, to guide more correctly the consensus reaching processes.

4. A *feedback mechanism* is defined using the above consensus criteria. It substitutes the moderator's actions, avoiding the possible subjectivity that he/she can introduce, and gives advice to the experts to find out the changes they need to make in their opinions in order to obtain the highest degree of consensus possible.

In particular, our consensus model develops its activity in three phases (Figure 4): *computing consensus degrees, controlling the consensus state* and *feedback mechanism.*

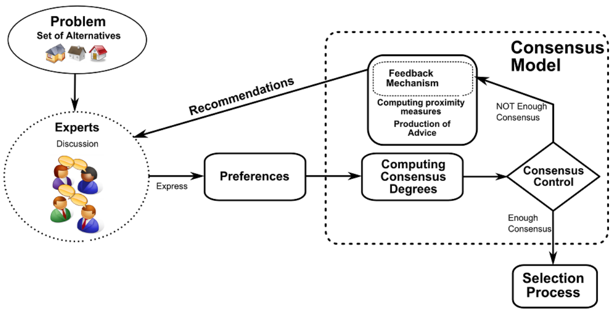

Figure 4: Consensus model

3.1 COMPUTING CONSENSUS DEGREES

The consensus degrees are used to measure the current level of consensus in the decision process. They are given at three different levels: pairs of alternatives, alternatives and relations. The computation of the consensus degrees is carried out as follows:

1. For each pair of experts, e_i, e_j $(i < j)$, a similarity matrix, $SM_{ij} = (sm_{ij}^{lk})$, is defined where

$$sm_{ij}^{lk} = 1 - \frac{|\Delta_{t'}^{-1}(TF_{t'}^{t}(p_i^{lk})) - \Delta_{t'}^{-1}(TF_{t'}^{t}(p_j^{lk}))|}{n(t') - 1}. \quad (5)$$

being $p_i^{lk} = (s_v^{n(t)}, \alpha_1)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$ and $p_j^{lk} = (s_w^{n(t)}, \alpha_2)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$.

2. A consensus matrix, CM, is calculated by aggregating all the similarity matrices using the arithmetic mean as the aggregation function ϕ:

$$cm^{lk} = \phi(sm_{ij}^{lk}). \quad (6)$$

3. Once the consensus matrix, CM, is computed, we proceed to calculate the consensus degrees at the three different levels:

   (a) **Level 1.** *Consensus degree on pairs of alternatives, cp^{lk}.* It measures the agreement on the pair of alternatives (x_l, x_k) amongst all the experts.

$$cp^{lk} = cm^{lk}. \quad (7)$$

   (b) **Level 2.** *Consensus degree on alternatives, ca^l.* It measures the agreement on an alternative x_l amongst all the experts.

$$ca^l = \frac{\sum_{k=1, k\neq l}^{n} cp^{lk}}{n-1}. \quad (8)$$

   (c) **Level 3.** *Consensus degree on the relation, cr.* It measures the global consensus degree amongst the experts' opinions.

$$cr = \frac{\sum_{l=1}^{n} ca^l}{n}. \quad (9)$$

3.2 CONTROLLING THE CONSENSUS STATE

The consensus state control process involves deciding when the consensus process should be finished. To do so, a minimum consensus threshold, $\gamma \in [0, 1]$, is fixed before applying the consensus model. When the consensus measure, cr, satisfies this value, γ, the consensus model finishes and the selection process is applied to obtain the solution. Otherwise, the feedback mechanism is applied. To avoid that the consensus process does not converge, a maximum number of consensus rounds, *MaxRounds*, is incorporated.

3.3 FEEDBACK MECHANISM

The feedback mechanism provides recommendations to support the experts in changing their opinions. It consists on two steps: *computation of proximity values* and *production of advice*, which are explained in detail in the following subsections.

3.3.1 Computation of Proximity Measures

These measures evaluate the agreement between the individual experts' opinions and the group opinion. To compute them for each expert, we need to obtain the collective unbalanced fuzzy linguistic preference relation, $P_{e_c} = (p_c^{lk})$, calculated by means of the aggregation of the set of individual unbalanced fuzzy linguistic preference relations $\{P_{e_1}, \ldots, P_{e_m}\}$ as follows

$$p_c^{lk} = \phi_{un}(p_1^{lk}, \ldots, p_m^{lk}). \quad (10)$$

with ϕ_{un} the $LOWA_{un}$ operator defined in section 2.2.2.

Once P_{e_c} is obtain, we can compute the proximity measures carrying out the following two steps:

1. For each expert, e_i, a proximity matrix, $PM_i = (pm_i^{lk})$, is obtained where

$$pm_i^{lk} = 1 - \frac{|\Delta_{t'}^{-1}(TF_{t'}^{t}(p_i^{lk})) - \Delta_{t'}^{-1}(TF_{t'}^{t}(p_c^{lk}))|}{n(t') - 1}. \quad (11)$$

being $p_i^{lk} = (s_v^{n(t)}, \alpha_1)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$ and $p_c^{lk} = (s_w^{n(t)}, \alpha_2)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$.

2. Computation of proximity measures at three different levels:

(a) **Level 1.** *Proximity measure on pairs of alternatives,* pp_i^{lk}*.* It measures the proximity between the preferences, on each pair of alternatives , of the expert, e_i, and the group.

$$pp_i^{lk} = pm_i^{lk}. \tag{12}$$

(b) **Level 2.** *Proximity measure on alternatives,* pa_i^l*.* It measures the proximity between the preferences, on each alternative, x_l, of the expert, e_i, and the group.

$$pa_i^l = \frac{\sum_{k=1,k\neq l}^{n} pp_i^{lk}}{n-1}. \tag{13}$$

(c) **Level 3.** *Proximity measure on the relation,* pr_i*.* It measures the global proximity between the preferences of each expert, e_i, and the group.

$$pr_i = \frac{\sum_{l=1}^{n} pa_i^l}{n}. \tag{14}$$

3.3.2 Production of Advice

The production of advice to achieve a solution with the highest degree of consensus possible is carried out in two steps: *Identification rules* and *Direction rules.*

1. **Identification rules (IR).** We must identify the experts, alternatives and pairs of alternatives that are contributing less to reach a high degree of consensus and, therefore, should participate in the change process.

   (a) *Identification rule of experts (IR.1).* It identifies the set of experts, $EXPCH$, that should receive advice on how to change some of their preference values.

   $$EXPCH = \{e_i \mid pr_i < \gamma\} \tag{15}$$

   (b) *Identification rule of alternatives (IR.2).* It identifies the alternatives whose associated assessments should be taken into account by the above experts in the change process of their preferences.

   $$ALT = \{x_l \in X \mid ca^l < \gamma\} \tag{16}$$

   (c) *Identification rule of pairs of alternatives (IR.3).* It identifies the particular pairs of alternatives (x_l, x_k) whose respective associated assessments p_i^{lk} the expert e_i should change.

   $$\begin{aligned} PALT_i = \{(x_l, x_k) \mid x_l \in ALT \wedge \\ e_i \in EXPCH \wedge pp_i^{lk} < \gamma\} \end{aligned} \tag{17}$$

2. **Direction rules (DR).** We must find out the direction of change to be applied to the preference assessment p_i^{lk}, with $(x_l, x_k) \in PALT_i$. To do this, we define the following two direction rules.

   (a) *DR.1.* If $p_i^{lk} > p_c^{lk}$, the expert e_i should decrease the assessment associated to the pair of alternatives (x_l, x_k), i.e., p_i^{lk}.

   (b) *DR.2.* If $p_i^{lk} < p_c^{lk}$, the expert e_i should increase the assessment associated to the pair of alternatives (x_l, x_k), i.e., p_i^{lk}.

4 Example of Application

Let us suppose that three different experts $E = \{e_1, e_2, e_3\}$ provide the following unbalanced fuzzy linguistic preference relations over a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$ using the unbalanced linguistic term set $S_{un} = \{N, VL, L, M, H, QH, VH, T\}$ (see Figure 1 and Figure 3):

$$P_{e_1} = \begin{pmatrix} - & (H,0) & (QH,0) & (L,0) \\ (VL,0) & - & (M,0) & (H,0) \\ (L,0) & (M,0) & - & (L,0) \\ (VH,0) & (VL,0) & (VH,0) & - \end{pmatrix}$$

$$P_{e_2} = \begin{pmatrix} - & (VL,0) & (L,0) & (VH,0) \\ (H,0) & - & (QH,0) & (T,0) \\ (QH,0) & (L,0) & - & (H,0) \\ (VL,0) & (N,0) & (L,0) & - \end{pmatrix}$$

$$P_{e_3} = \begin{pmatrix} - & (VL,0) & (M,0) & (VH,0) \\ (H,0) & - & (QH,0) & (L,0) \\ (M,0) & (L,0) & - & (T,0) \\ (L,0) & (H,0) & (N,0) & - \end{pmatrix}$$

FIRST ROUND

In the following, we show how to apply each step of the consensus model.

1. **Computing consensus degrees:**

   (a) *Similarity matrices:*

$$SM_{12} = \begin{pmatrix} - & 0.50 & 0.50 & 0.37 \\ 0.50 & - & 0.75 & 0.62 \\ 0.50 & 0.75 & - & 0.62 \\ 0.25 & 0.87 & 0.37 & - \end{pmatrix}$$

$$SM_{13} = \begin{pmatrix} - & 0.50 & 0.75 & 0.37 \\ 0.50 & - & 0.75 & 0.62 \\ 0.75 & 0.75 & - & 0.25 \\ 0.37 & 0.50 & 0.12 & - \end{pmatrix}$$

$$SM_{23} = \begin{pmatrix} - & 1.00 & 0.75 & 1.00 \\ 1.00 & - & 1.00 & 0.25 \\ 0.75 & 1.00 & - & 0.62 \\ 0.87 & 0.37 & 0.75 & - \end{pmatrix}$$

(b) *Consensus matrix:*

$$CM = \begin{pmatrix} - & 0.66 & 0.66 & 0.58 \\ 0.66 & - & 0.83 & 0.49 \\ 0.66 & 0.83 & - & 0.49 \\ 0.49 & 0.58 & 0.41 & - \end{pmatrix}$$

(c) *Consensus degrees on pairs of alternatives.* The element (l, k) of CM represents the consensus degrees on the pair of alternatives (x_l, x_k).

(d) *Consensus on alternatives:*

$$ca^1 = 0.63 \quad ca^2 = 0.66 \quad ca^3 = 0.66 \quad ca^4 = 0.49$$

(e) *Consensus on the relation:*

$$cr = 0.61$$

2. **Controlling the consensus state:** In this example, we have decided to use the value, $\gamma = 0.75$. Because $cr < \gamma$, then it is concluded that there is no consensus amongst the experts, and consequently, the consensus model computes the proximity measures to support the experts on the necessary changes in their preferences in order to increase cr.

3. **Feedback mechanism:** To calculate P_{e_c}, the systems uses the $LOWA_{un}$ operator and the linguistic quantifier *most of* defined as $Q(r) = r^{1/2}$, which applying (4), generates the following weighting vector $W = \{0.58, 0.24, 0.18\}$.

$$P_{e_c} = \begin{pmatrix} - & (M, -0.34) & (H, -0.20) & (QH, 0.10) \\ (M, 0.28) & - & (QH, -0.37) & (QH, 0.20) \\ (H, -0.20) & (M, -0.42) & - & (QH, 0.20) \\ (H, -0.28) & (M, -0.43) & (H, -0.46) & - \end{pmatrix}$$

(a) **Computation of proximity measures.**

i. *Proximity matrices:*

$$PM_1 = \begin{pmatrix} - & 0.83 & 0.85 & 0.48 \\ 0.59 & - & 0.79 & 0.85 \\ 0.65 & 0.95 & - & 0.47 \\ 0.71 & 0.68 & 0.69 & - \end{pmatrix}$$

$$PM_2 = \begin{pmatrix} - & 0.67 & 0.65 & 0.89 \\ 0.91 & - & 0.95 & 0.77 \\ 0.85 & 0.80 & - & 0.85 \\ 0.53 & 0.55 & 0.68 & - \end{pmatrix}$$

$$PM_3 = \begin{pmatrix} - & 0.66 & 0.90 & 0.89 \\ 0.91 & - & 0.95 & 0.47 \\ 0.90 & 0.80 & - & 0.77 \\ 0.66 & 0.82 & 0.43 & - \end{pmatrix}$$

ii. *Proximity on pairs of alternatives for expert e_i are given in PM_i.*

iii. *Proximity on alternatives (See Table 1):*

Table 1: Proximity measures on alternatives

x_1	x_2	x_3	x_4
$pa_1^1 = 0.72$	$pa_1^2 = 0.74$	$pa_1^3 = 0.69$	$pa_1^4 = 0.69$
$pa_2^1 = 0.73$	$pa_2^2 = 0.87$	$pa_2^3 = 0.83$	$pa_2^4 = 0.58$
$pa_3^1 = 0.82$	$pa_3^2 = 0.78$	$pa_3^3 = 0.82$	$pa_3^4 = 0.64$

iv. *Proximity on the relation:*

$$pr_1 = 0.71 \quad pr_2 = 0.75 \quad pr_3 = 0.76$$

(b) **Production of advice.**

i. *Identification rules.*
(IR.1) Set of experts to change their preferences, $EXPCH$:

$$EXPCH = \{e_i \mid pr_i < 0.75\} = \{e_1\}$$

(IR.2) Set of alternatives whose assessments should be considered in the change process, ALT:

$$ALT = \{x_l \in X \mid ca^l < 0.75\} = \{x_1, x_2, x_3, x_4\}$$

(IR.3) Set of pairs of alternatives whose associated assessments should change, $PALT_i$:

$$PALT_1 = \{(x_1, x_4), (x_2, x_1), (x_3, x_1), (x_3, x_4), (x_4, x_1), (x_4, x_2), (x_4, x_3)\}$$

which gives the following list of preference values:

$$p_1^{14} \quad p_1^{21} \quad p_1^{31} \quad p_1^{34} \quad p_1^{41} \quad p_1^{42} \quad p_1^{43}$$

ii. *Direction rules.*
Because $p_1^{14} < p_c^{14}$, $p_1^{21} < p_c^{21}$, $p_1^{31} < p_c^{31}$, $p_1^{34} < p_c^{34}$, $p_1^{42} < p_c^{42}$ and $p_1^{41} > p_c^{41}$, $p_1^{43} > p_c^{43}$ expert e_1 is advised to increase the assessment of the first five preference values and decrease the assessment of the last two preference values.

SECOND ROUND

1. **Providing new preferences:** In this example, we suppose that expert e_1 follows the advice given, and thus, his/her new preferences is as follows:

$$P_{e_1} = \begin{pmatrix} - & (H, 0) & (QH, 0) & (\mathbf{VH}, \mathbf{0}) \\ (\mathbf{M}, \mathbf{0}) & - & (M, 0) & (H, 0) \\ (\mathbf{M}, \mathbf{0}) & (M, 0) & - & (\mathbf{H}, \mathbf{0}) \\ (\mathbf{VL}, \mathbf{0}) & (\mathbf{M}, \mathbf{0}) & (\mathbf{L}, \mathbf{0}) & - \end{pmatrix}$$

2. **Computing consensus degrees.**

(a) *Similarity matrices:*

$$SM_{12} = \begin{pmatrix} - & 0.50 & 0.50 & 1.00 \\ 0.87 & - & 0.75 & 0.62 \\ 0.75 & 0.75 & - & 0.62 \\ 1.00 & 0.50 & 0.50 & - \end{pmatrix}$$

$$SM_{13} = \begin{pmatrix} - & 0.50 & 0.75 & 1.00 \\ 0.87 & - & 0.75 & 0.62 \\ 1.00 & 0.75 & - & 0.62 \\ 0.87 & 0.87 & 0.25 & - \end{pmatrix}$$

$$SM_{23} = \begin{pmatrix} - & 1.00 & 0.75 & 1.00 \\ 1.00 & - & 1.00 & 0.25 \\ 0.75 & 1.00 & - & 0.62 \\ 0.87 & 0.37 & 0.75 & - \end{pmatrix}$$

(b) *Consensus matrix:*

$$CM = \begin{pmatrix} - & 0.66 & 0.66 & 1.00 \\ 0.91 & - & 0.83 & 0.50 \\ 0.83 & 0.83 & - & 0.62 \\ 0.91 & 0.58 & 0.83 & - \end{pmatrix}$$

(c) *Consensus degrees on pairs of alternatives.* The element (l, k) of CM represents the consensus degrees on the pair of alternatives (x_l, x_k).

(d) *Consensus on alternatives:*

$$ca^1 = 0.77 \quad ca^2 = 0.75 \quad ca^3 = 0.76 \quad ca^4 = 0.77$$

(e) *Consensus on the relation:*

$$cr = 0.76$$

3. **Controlling the consensus state:** As we can observe, the changes in the preference values introduced result in an increasing of the global consensus from 0.61 to 0.76. The minimum consensus threshold is reached, $cr = 0.76 > \gamma = 0.75$, and, therefore, the consensus model would stop and the selection process would be applied to obtain the final solution of consensus.

5 CONCLUDING REMARKS

In this paper we have proposed a consensus model for GDM problems with unbalanced fuzzy linguistic information which allows to manage consensus processes based on similarity measures among preferences and to build recommendation systems of preferences to support the consensus process automatically, without moderator.

Acknowledgements

This work has been supported by the Research Projects TIN2007-61079 and SAINFOWEB-PAI00602.

References

[1] J.L. García-Lapresta. A general class of simple majority decision rules based on linguistic opinions. *Information Sciences* **176:4** (2006) 352-365.

[2] F. Herrera, E. Herrera-Viedma, L. Martinez. A Hierarchical Ordinal Model for Managing Unbalanced Linguistic Term Sets based on the Linguistic 2-tuple Model. *Eurofuse Workshop on Preference Modelling and Applications*, Granada (Spain), April 2001, 201-206.

[3] F. Herrera, E. Herrera-Viedma and J.L. Verdegay. Direct approach processes in group decision making using linguistic OWA operators. *Fuzzy Sets and Systems* **79** (1996) 175-190.

[4] F. Herrera, E. Herrera-Viedma and J.L. Verdegay: A model of consensus in group decision making under linguistic assessments. *Fuzzy Sets and Systems* **78** (1996) 73-87.

[5] F. Herrera and L. Martínez. A 2–tuple fuzzy linguistic representation model for computing with words. *IEEE Transaction on Fuzzy Systems* **8:6** (2000) 746-752.

[6] F. Herrera and L. Martínez. A model based on linguistic 2-tuples for dealing with multigranularity hierarchical linguistic contexts in multiexpert decision-making. *IEEE Trans. on Sys., Man and Cyb. Part B: Cyb.* **31:2** (2001) 227-234.

[7] E. Herrera-Viedma and A.G. López-Herrera. A model of information retrieval system with unbalanced fuzzy linguistic information. *International Journal of Intelligent Systems* **22:11** (2007) 1197-1214.

[8] J. Kacprzyk, H. Nurmi and M. Fedrizzi. *Consensus under fuzziness* (Boston: Kluwer Academic Publishers, 1997).

[9] V. Torra. Aggregation of linguistic labels when semantics is based on antonyms. *International Journal of Intelligent Systems* **16** (2001) 513-524.

[10] Z.S. Xu. A method for multiple attribute decision making with incomplete weight information in linguistic setting. *Knowledge-Based Systems* **20:8** (2007) 719-725.

[11] R.R. Yager. On ordered weighted averaging aggregation operators in multicriteria decision making. *IEEE Trans. on Sys., Man and Cyb. Part B: Cyb.* **10:1** (1988) 183-190.

[12] L.A. Zadeh. A computational approach to fuzzy quantifiers in natural language. *Computation & Mathematics with Applications* **9** (1983) 149-184.